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Effective descriptive complexity of some compact sets

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ABSTRACT

Investigating the descriptive complexity of sets has been a topic of theoretical computer science, particularly general recursion theory. In this paper, we investigate the effective descriptive complexity of certain sets of real numbers, namely, compact sets corresponding to a countable collection of closed intervals under the standard euclidian topology. We derive the result that the descriptive complexity of defining such sets is Π_1^1, which corresponds to the descriptive complexity of well-orderings. To prove such a result, we define several functions and relations that could be effectively computed by some abstract computing machine with access to pre-loaded infinite inputs.

INTRODUCTION

In theoretical computer science (Moschovakis 2009), (Chong and Yu 2015) and descriptive set theory (Kechris 2012), (Hinman 2017), (Yu 2020), descriptive complexity refers to the logical complexity involved in defining a set, particularly sets of natural numbers or sets of real numbers. Intuitively, a set is more complex descriptively if the logical formula that defines it requires a longer sequence of alternating universal and existential quantifiers (Moschovakis 2010). Descriptive complexities of various sets form a well-defined strict hierarchy corresponding to the arithmetical and analytical hierarchies. For instance, the simplest sets to describe belong to Δ_0^0 - which is the lowest rank of the arithmetical hierarchy and corresponds to sets of numbers whose membership can be computed using recursive functions. Above this rank lie sets belonging to Σ_1^0, which correspond to sets of numbers whose membership can be practically computed by some computer (e.g., a Turing Machine) - albeit without guarantees of halting (Forster et al. 2020). More complex than sets belonging to the arithmetical hierarchy however are sets that belong to the analytical hierarchy, where quantifiers are no longer over mere numbers, but over elements of the Baire Space (ω^ω), i.e., functions from numbers to numbers representing the reals (Hinman 2017).

The subject of this paper is similar in spirit to that of (Lutzen 1985) which studied the descriptive complexity of function spaces or that of (Matheron 1995) which studied the descriptive complexity of Helson Sets. Within the area of mathematics, the results of our study would provide insights on the relationships between topological spaces or whether such relationships have effective mappings. Such questions are analogous to the study of functors that map various mathematical spaces as done in (Herrlich 1974). Within the area of computer science, particularly theoretical computer science, a pioneering work on the mathematical foundations of programming language semantics is from (Scott 1981), whereby interest in topology and computer science (particularly recursion theory) developed under the result that “open sets are semidecidable properties” (Spreen 1990). In this paper, we investigate the descriptive complexity of a specific type of set formed from the real numbers. Namely, we investigate sets that are countable and are formed from unions of closed intervals of real numbers under the euclidian topology, and which are compact, i.e., any open covering of these sets have a finite subcovering.

Our result is that the descriptive complexity of such sets belong to the Π_1^1 level of the hierarchy. However, to come up with an effective description, we first construct a mapping from the collection of such sets, to functions in ω^ω. This is an important point to consider given that sets in Π_1^1 are defined effectively using some recursive relations with elements belonging to either ω or ω^ω.

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PRELIMINARIES

Using standard definitions from general topology (Munkres 2000), given a set X, a topological space is a structure $(X, \mathcal{T})$, where $\mathcal{T}$ is a collection of subsets of X defined as *open* sets such that $\mathcal{T}$ is closed under finite intersections and countable unions. The collection $\mathcal{T}$ is a *topology* for X. Complements of open sets are the *closed* sets. A *basis* for $\mathcal{T}$ is a collection of sets $\mathcal{B} \subseteq \mathcal{T}$ such that for each $U \in \mathcal{T}$, there exists a $B \in \mathcal{B}$ such that $B \subseteq U$. Moreover, given any $U_1, U_2 \in \mathcal{T}$, there exists a $B \in \mathcal{B}$ such that $B \in U_1 \cap U_2$. If there exists a basis $\mathcal{B}$ for $\mathcal{T}$ that is countable, $\mathcal{T}$ is termed *separable*.

If X is the set of real numbers, a standard topology over X has as its basis the collection of all open intervals, i.e., each basis element of $\mathcal{T}$ is an open interval (a, b) (for some real numbers a, b), whereby $x \in (a, b) \leftrightarrow a < x < b$. It follows that the closed sets in this case are the closed intervals $[a, b]$ for some real numbers a, b such that $x \in [a, b] \leftrightarrow a \leq x \leq b$. This particular topological space with open intervals as basis is also known as the *euclidian* topology given that it is induced by the euclidian metric over X. It can be shown that $(X, \mathcal{T})$ in this case is separable, whose basis consists of all open intervals with rational endpoints.

Compactness

Given $\mathcal{O} \subseteq \mathcal{T}$, $\mathcal{O}$ *covers* $A \subseteq X$ if and only if for each $x \in A$, there exists an open set $U \in \mathcal{O}$ such that $x \in U$. Given a collection of sets $\mathcal{A}$, we have that $\mathcal{O}$ covers $\mathcal{A}$ if and only if $\mathcal{O}$ covers $\bigcup \mathcal{A}$. In set theory however, for any collection of sets $\mathcal{A}$, the union $\bigcup \mathcal{A}$ is also a set, so that the two notions of coverings mentioned can be used analogously. Using the notion of coverings, the following definition states the notion of compactness (Munkres 2000).

Definition 1 *Given a topological space $(X, \mathcal{T})$, a set $A \subseteq X$ is compact if and only if for any $\mathcal{O} \subseteq \mathcal{T}$ that covers A, there exists a finite subcover $\mathcal{F} \subseteq \mathcal{O}$ that also covers A.*

Definition 2 *Given a topological space $(X, \mathcal{T})$, a collection of sets $\mathcal{A}$ with elements $A \subseteq X$ is compact if and only if $\bigcup \mathcal{A}$ is compact.*

Once again, given that in set theory, for any collection of sets $\mathcal{A}$, the union $\bigcup \mathcal{A}$ is also a set, the two notions of compactness above are used analogously throughout the paper. For instance, if $\mathcal{O}$ covers $\mathcal{A}$, it is understood that each $A \in \mathcal{A}$ is covered by some $\mathcal{F} \subseteq \mathcal{O}$ such that $A \subseteq \bigcup \mathcal{F}$.

Well Orderings

Let X be as set, and let $Z \subseteq X \times X$. Z is a well-ordering if and only if it is reflexive, antisymmetric, connected, transitive and well-founded. In particular, Z is well-founded if and only if each subset of X has a least element with respect to Z, i.e., $(\forall S \subseteq X)[S \neq \emptyset \rightarrow (\exists m \in S)(\forall s \in S)\neg(sZm)]$. The notion of well-foundedness could also be expressed as $\forall \phi[\forall m(\phi(m+1), \phi(m)) \in Z \rightarrow \exists m(\phi(m), \phi(m+1)) \in Z]$ (where ϕ is a function in ω^ω). This definition of well-foundedness however requires the Axiom of Dependent Choice (Hinman 2017). Throughout the paper, the class of all well-orderings is referred to as W.

Recursion Theoretic Concepts

Let elements of ω be written using small alphabetic letters, i.e., n, k, l, etc. and let elements of ω^ω (i.e., functions from ω to ω) be written using greek letters, i.e., α, β, etc. For some function α and integer $n \geq 0$, let $\overline{\alpha}(n)$ define the course-of-values of α from 0 up to $n-1$, i.e., $\overline{\alpha}(n) = \{\alpha(0), \alpha(1), \ldots, \alpha(n-1)\}$.

For some $k, l \geq 0$, let $R^{k,l}(\mathbf{m}, \boldsymbol{\alpha})$ define a relation where $\mathbf{m} \in \omega^k$ and $\boldsymbol{\alpha} \in (\omega^\omega)^l$. For ease of notation, the superscripts k, l are usually omitted from R if the values of k and l are understood. By definition, $R^{k,l}$ is a relation involving both number inputs and function inputs, i.e., elements of ω and ω^ω. Inputs in the form of functions may not seem to be practical from the perspective of a computer given that the amount of time needed to feed such an input would be infinite. However, following (Hinman 2017), the computer in this case is assumed to be connected to some device with infinite memory such that the function input is pre-loaded before computation starts. Since input loading is not considered in the computation, this allows the computer to still perform finite computations despite infinitely-sized inputs.

Basic Recursion Theoretic Hierarchies

Following standard definitions on the arithmetical hierarchy, we have $R \in \Delta_0^0$ for the simplest case where R is recursive. From (Hinman 2017), R is recursive if and only if the characteristic function for R is a recursive functional, i.e., functions belonging to the set of primitive recursion functions and closed under both bounded and unbounded recursion and functional composition. $R \in \Sigma_1^0$ if R is semi-recursive, i.e., $R(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists n P(n, \mathbf{m}, \boldsymbol{\alpha})$ for some $P \in \Delta_0^0$. Complements of Σ_1^0 are defined as Π_1^0 (i.e., the non semi-recursive relations). In general, for any $n \geq 0$, we have $R(\mathbf{m}, \boldsymbol{\alpha}) \in \Sigma_n^0 \leftrightarrow \exists n P(n, \mathbf{m}, \boldsymbol{\alpha})$ for some $P \in \Delta_{n-1}^0$ and Π_n^0 as the complement of Σ_n^0. Relations in Δ_n^0 are those in $\Sigma_n^0 \cap \Pi_n^0$. It is a standard result of recursion theory that for any $n \geq 1$, we have $\Delta_{n-1}^0 \subsetneqq \Sigma_n^0 \cup \Pi_n^0$.

Above the arithmetical hierarchy lies the analytical hierarchy defined using symbols $\Delta_n^1, \Sigma_n^1, \Pi_n^1$ for $n \geq 0$. A relation $R(\mathbf{m}, \boldsymbol{\alpha}) \in \Sigma_1^1$, just in case $R(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists \beta \forall n P(n, \mathbf{m}, \boldsymbol{\alpha}, \beta)$ for some arithmetical P, (i.e., in the analytical hierarchy, instead of quantifying over numbers, quantifiers are over functions which are elements of ω^ω). The complement of Σ_1^1 is Π_1^1 so that $R(\mathbf{m}, \boldsymbol{\alpha}) \in \Pi_1^1 \leftrightarrow \forall \beta \exists n P(n, \mathbf{m}, \boldsymbol{\alpha}, \beta)$ for some arithmetical P. In general, given $n \geq 0$, we have Π_n^1 as the complement of Σ_n^1 and Δ_n^1 as relations in $\Sigma_n^1 \cap \Pi_n^1$. Similar to the arithmetical hierarchy, a standard result in general recursion theory is that for any $n \geq 1$, we have $\Delta_{n-1}^1 \subsetneqq \Sigma_n^1 \cup \Pi_n^1$.

A standard result in the analytical hierarchy is *quantifier contraction* (Hinman 2017). That is, any logical formula of the form $\forall \beta \exists a \forall b P(\beta, a, b)$ for some P could be equivalently expressed as $\forall \beta \exists z P'(\beta, z)$ for some P'. Similarly, any logical formula of the form $\exists \beta \forall a \exists b P(\beta, a, b)$ could be equivalently expressed as $\exists \beta \forall z P'(\beta, z)$ for some P' and z.

Reductions to Well Orderings

A relation R is *many-one* reducible to a set A (i.e., $R \ll A$) if and only if for some recursive function F, we have $R(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow F(\mathbf{m}, \boldsymbol{\alpha}) \in A$. It follows that if $A \in \Sigma_r^1$ for some r, and $R \ll A$, then also $R \in \Sigma_r^1$. The following is a well-known result in general recursion theory, which essentially states that any Π_1^1 relation is many-one reducible to some well-ordering.

Theorem 1 *For any R, $R \in \Pi_1^1 \leftrightarrow R \ll W$.*

Proof. See (Hinman 2017). ■

RESULTS ON SOME COMPACT SPACES

Throughout this section, let X be the set of real numbers, and let $(X, \mathcal{T})$ be the euclidian topological space (which is separable). As stated previously, elements of $\mathcal{T}$ are the open sets whose basis consists of open intervals (a, b) for some $a, b \in \mathbb{Q}$. The closed intervals in turn are written as $[c, d]$ for some $c, d \in X$. It follows that each open (and likewise closed) interval is completely determined by its endpoints and there is a one-to-one and onto mapping between the set of open (and likewise closed) intervals to $X \times X$. Viewing open and closed intervals in terms of tuples in $X \times X$ aids in constructing effective procedures. Thus, throughout this section, an open / closed interval is identified with its endpoints. For instance, if a function f has $\mathcal{V}$ as input (for some collection of open intervals $\mathcal{V}$), it is understood that $\mathcal{V} = \{(a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n)\}$ for some n, and the domain of f is identified with $P(X \times X)$.

Definition 3 *Let $\mathcal{V}$ be a countable collection of closed intervals with rational endpoints and let V be the collection of all such $\mathcal{V}$.*

It follows that there is a one-to-one and onto mapping between V and $P(\mathbb{Q} \times \mathbb{Q})$. For the rest of this paper, it is assumed that the notation $\mathcal{V}$ refers to this special type of collection of sets. Functions that have $\mathcal{V}$ as input are assumed to operate on a collection of tuples (each of length 2) representing the rational endpoints of closed intervals contained in $\mathcal{V}$.

Mappings from Sets to Functions

The task of this section is to evaluate the descriptive complexity of compact $\mathcal{V} \in$ V. However, given that sets belonging to the arithmetic and analytical hierarchy have effective descriptions, the first step needed is to construct a mapping such that any $\mathcal{V} \in$ V can be described by some function in $[0,1]^\omega \subseteq \omega^\omega$. These mappings are f and g described in Lemmas 1 and 2 below.

Lemma 1 *There exists an injective function $f: V \to \{0,1\}^\omega$.*

Proof. Let $\mathcal{V} \in$ V. Let h be any one-to-one and onto mapping from the set of rational numbers to ω (such a mapping exists since both the rationals and ω are countable). Let p be a pairing function from $\omega \times \omega$ to ω. Let $\mathcal{V}(n)$ refer to the nth element of $\mathcal{V}$, i.e., a closed interval $[a_n, b_n]$ whose endpoints are a_n and b_n ($\mathcal{V}$ can be indexed by n since it is countable by definition). Let $\mathcal{V}(n)_a = a_n$ and $\mathcal{V}(n)_b = b_n$. We define $f(\mathcal{V})$ as follows for any $n \geq 0$ such that $\mathcal{V}(n)$ is defined.

$$f(\mathcal{V})(p(h(\mathcal{V}(n)_a), h(\mathcal{V}(n)_b))) = 1$$

For all other coordinates (say m) of $f(\mathcal{V})$ such that m does not equal the value of $p(h(\mathcal{V}(n)_a), h(\mathcal{V}(n)_b))$ for any n, then $f(\mathcal{V})(m) = 0$. ■

Lemma 2 *Let $\mathcal{B}$ be any collection of basis elements, and let B be the collection of all such $\mathcal{B}$. There exists an injective function $g: B \to [0,1]^\omega$.*

Proof. As $(X, \mathcal{T})$ is separable, let $G_1, G_2, \ldots$ be an enumeration of all basis elements of $\mathcal{T}$. Each element of $\mathcal{B}$ corresponds to an open interval $G_n = (a_n, b_n)$ for some n. Thus, we can define g using a similar function as the f described in the previous Lemma. ■

Example 1 For a very simple example of Lemma 1, without loss of generality, let $\mathcal{V} \in$ V be a singleton composed of the single closed unit interval $\mathcal{V} = \{[0,1]\}$, so that $\mathcal{V}(0) = [0,1]$ and $\mathcal{V}$(n) is undefined for $n > 0$. For the sake of this example, let $\omega = \mathbb{N}$. To construct $h: \mathbb{Q} \to \mathbb{N}$, first let $q^+: \mathbb{Q}^+ \to \mathbb{Z}^+$ be the Cantor bijection from the set of positive rationals to positive integers, i.e., $q^+(1) = 1, q^+(1/2) = 3, q^+(2) = 2, q^+(3) = 4, q^+(1/3) = 5$, etc. Let $q: \mathbb{Q} \to \mathbb{Z}$ be:

$$q(x)\begin{cases} q^+(x) \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -q^+(-x) \text{ if } x < 0 \end{cases}$$

Let $r: \mathbb{Z} \to \mathbb{N}$ be $r(k) = 2k$ if $k > 0$ and $r(k) = -2k + 1$ if $k \leq 0$. Given q and r define h as $h \coloneqq r \circ q$. Let $p: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the usual Cantor pairing function, i.e.,

$$p(n, m) = \frac{1}{2}(n + m - 2)(n + m - 1) + m$$

Using the definition for f in Lemma 1, we have $f(\{[0,1]\})$ as an infinite sequence of zeros and ones such that $f(\{[0,1]\})(3) = 1$ (since we have $n = h(0) = r(q(0)) = 1$ and $m = h(1) = r(q(1)) = 2$ in the definition of p above). All other coordinates of $f(\{[0,1]\})$ are zero. If instead we have $\mathcal{V} = \{[0,1], [2,3]\}$ then $\mathcal{V}(0) = [0,1]$, $\mathcal{V}(1) = [2,3]$, and $\mathcal{V}$(n) is undefined for $n > 1$. It follows that $f(\{[0,1], [2,3]\})(3) = 1$ and $f(\{[0,1], [2,3]\})(63) = 1$ (since for $\mathcal{V}(1) = [2,3]$, we have $n = h(2) = r(q(2)) = 4$ and $m = h(3) = r(q(3)) = 8$ in the definition of p above). The same analogously holds for g, except that instead of closed intervals, $\mathcal{B}$ is a set of basis elements.

Definition 4 *Let f and g be the functions described in Lemmas 1 and 2 respectively. Given $\mathcal{V} \in V$ and $\mathcal{B} \in B$, let $\alpha = f(\mathcal{V})$ and $\beta = g(\mathcal{B})$. Given n, the set that corresponds to $\overline{\alpha}(n)$ is:*

$$\{f^{-1}(\alpha(i_0)), f^{-1}(\alpha(i_1)), \ldots, f^{-1}(\alpha(i_k))\}$$

for some k with $0 \leq k \leq n - 1$ such that for $0 \leq j \leq k$, we have $\alpha(i_j) = 1$. It follows that each $f^{-1}(\alpha(i))$ for some i is a closed interval contained in $\mathcal{V}$. Similarly, given m, the set that *corresponds to* $\overline{\beta}(m)$ is:

$$\{g^{-1}(\beta(i_0)), g^{-1}(\beta(i_1)), \ldots, g^{-1}(\beta(i_l))\}$$

for some l with $0 \leq l \leq m - 1$ such that for $0 \leq j \leq l$, we have $\beta(i_j) = 1$. It follows that each $g^{-1}(\beta(i))$ corresponds to a basis element contained in $\mathcal{B}$.

Example 2 Let f be the same f constructed in Example 1. Once again, suppose that $\mathcal{V} = \{[0,1]\}$. From Example 1, we have $\alpha = f(\mathcal{V})$ as as an infinite sequence of zeros and ones such that $\alpha(3) = 1$ and zero in all other coordinates. Suppose $n = 4$. It follows that the set that corresponds to $\overline{\alpha}(4)$ is simply $\{[0,1]\}$. If $\mathcal{V} = \{[0,1], [2,3]\}$, then $\alpha(3) = 1$ and $\alpha(63) = 1$, and α is zero in all other coordinates. In this case, the set that corresponds to $\overline{\alpha}(4)$ is $\{[0,1]\}$ as before. But the set that corresponds to say, $\overline{\alpha}(100)$ is $\{[0,1], [2,3]\}$. The same analogously holds for g.

Effective Computation of Coverings

Having defined the mappings f and g in Lemmas, the next step is to define a recursive relation that would form part of the logical statement expressing the descriptive complexity of compact $\mathcal{V}$. This recursive relation is P described in Lemma 3. Throughout this section, both $\mathcal{B}$ and B are the same $\mathcal{B}$ and B as in Lemma 2.

Lemma 3 *Given any $\mathcal{V} \in V$ and $\mathcal{B} \in B$, let $\alpha = f(\mathcal{V})$ and $\beta = g(\mathcal{B})$. There exists a recursive relation P such that $P(n, m, \alpha, \beta)$ if and only if the collection of basis elements* *corresponding to $\overline{\beta}(m)$ covers the union of at most n closed intervals of $\mathcal{V}$, corresponding to $\overline{\alpha}(n)$.*

Proof. Let f and g be the functions described in Lemma 1 and 2 respectively. Fix n and m. Define the recursive function $\overline{g}: \omega^\omega \times \omega \to [0,1]^{<\omega}$ such that $\overline{g}(\beta, m) = \overline{\beta}(m)$. Similarly, define the recursive function $\overline{f}: \omega^\omega \times \omega \to [0,1]^{<\omega}$ such that given $\mathcal{V} \in \mathrm{V}$ and n, we have $\overline{f}(\alpha, n) = \overline{\alpha}(n)$. The sequence $\overline{\beta}(m)$ (and analogously, $\overline{\alpha}(n)$) is a finite sequence of 0's and 1's of length m such that for $0 \le i < m$, we have $\overline{\beta}(m)(i) = 1$ if and only if for some k, there is a basis element $(a_k, b_k) \in \mathcal{B}$ such that $g(\mathcal{B})(k) = p(h(\mathcal{B}(k)_a, \mathcal{B}(k)_b)) = i$. Similarly, for $0 \le j < n$ we have $\overline{\alpha}(n)(j) = 1$ if and only if for some l, there is a closed interval $[a_l, b_l] \in \mathcal{V}$ such that $f(\mathcal{V})(l) = p(h(\mathcal{V}(l)_a, \mathcal{V}(l)_b)) = j$. Let c be the total number of coordinates of $\overline{\alpha}(n)$ with value 1, and let d be the total number of coordinates of $\overline{\beta}(m)$ with value 1. Let $C_0, C_1, \ldots, C_c$ be the corresponding closed intervals for $\overline{\alpha}(n)$ (i.e., the closed intervals mapped to coordinates of $\overline{\alpha}(n)$ with value 1 as described above). Similarly, let $B_0, B_1, \ldots, B_d$ be the corresponding basis elements for $\overline{\beta}(m)$. To effectively compute P given parameters n, m, α, and β, first compute $\overline{\alpha}(n) = \overline{f}(\alpha, n)$ and $\overline{\beta}(m) = \overline{g}(\beta, m)$. Both $\overline{\alpha}(n)$ and $\overline{\beta}(m)$ are recursively computed. Afterwards, check each C_j for $0 \le j \le c < n$ if it is covered by any of the B_l for $0 \le l \le d < m$. This is a recursive computation given that it involves making comparisons among endpoints, i.e., $[a,b]$ is covered by (c,d) if and only if $c < a$ and $b < d$. If all C_j's are covered by at least one B_l, then the collection of basis elements corresponding to $\overline{\beta}(m)$ covers the union of at most n closed intervals of $\mathcal{V}$. This is a recursive computation given that all computations are recursive and the number of times they are performed is finite - being bounded by parameters $c < n$ and $d < m$. By construction of P, the biconditional implication of the Lemma is proven. ∎

Example 3 Continuing Examples 1 and 2, let $f: V \to \{0,1\}^\omega$ be the same as Example 1, and let $g: B \to [0,1]^\omega$ be the natural modification of f to a domain that is the collection of sets of basis elements. Once again, suppose that $\mathcal{V} = \{[0,1]\}$ is the singleton set composed of the unit interval. Suppose that $\mathcal{B} \in B$ is the set of basis elements of the form $(-1/2, 1/2 + n)$ for integers $n \ge 0$. As shown in Example 2, we have $\alpha(3) = 1$ and by a similar computation, we have for instance, $\beta(72) = 1$ (since for the basis element $(-1/2, 1/2)$ we have $h(-1/2) = r(q(-1/2)) = 7$ and $h(1/2) = r(q(1/2)) = 6$) and $\beta(x) = 0$ for all $x < 72$. Of course, an infinite number of coordinates of β are 1 since it is an infinite set of basis elements. Suppose that $n = 1$ and $m = 1$. It follows that $\overline{\alpha}(1)$ and $\overline{\beta}(1)$ are both empty so that $P(1,1,\alpha,\beta)$ holds vacuously. However, if $n = 4$ and $m = 3$, then $P(4,3,\alpha,\beta)$ is false since the set corresponding to $\overline{\alpha}(1)$ is set $\{[0,1]\}$ as shown in Example 2 (i.e., $C_0 = [0,1]$ according to the notation of the proof for Lemma 3), whereas the set corresponding to $\overline{\beta}(3)$ is empty. Likewise, $P(4,73,\alpha,\beta)$ is false since $C_0 = [0,1]$ but $B_0 = (-1/2, 1/2)$ and $(-1/2, 1/2)$ does not cover $[0,1]$. But for $n = 4$ and $m = 205$, $P(3,205,\alpha,\beta)$ holds - since for the basis element $(-1/2, 3/2)$ we have $h(-1/2) = r(q(-1/2)) = 7$ and $h(3/2) = r(q(3/2)) = 14$ so that $\beta(204) = 1$ and $\overline{\beta}(205)$ corresponds to the set $\{(-1/2, 1/2), (-1/2, 3/2)\}$ which covers $[0,1]$.

Descriptive Complexity Results

This section lays down the stated result of the paper, which is that the complexity of evaluating whether a set $\mathcal{V}$ is compact is not harder than defining a set in the Π^1_1 level of the analytical hierarchy. However, defining the descriptive complexity of a set involves the use of recursive relations whose elements belong to either ω or ω^ω. To simplify the analysis, we assume that all elements of the recursive relations are values of the functions f and g described in Lemmas 1 and 2. Computation of f and g are not factored in the evaluation of complexity as they are infinite in nature. Rather, it is assumed that given $\mathcal{V}$, computation of $\alpha = f(\mathcal{V})$ is done beforehand and α is loaded into the infinite memory of the computer before it starts computation. The same holds true for $\mathcal{B}$ and g, i.e., $\beta = g(\mathcal{B})$ is loaded into the infinite memory of the computer prior to any computation by the computer. Having described informally the abstract computational model, we now state the main theorem and corollaries of this paper.

Theorem 2 *Given any $\mathcal{V} \in V$, let $\alpha = f(\mathcal{V})$. There exists a relation $R^{0,1} \in \Pi^1_1$ such that any covering of $\mathcal{V}$ by basis elements of $\mathcal{T}$ has a finite subcover if and only if $R^{0,1}(\alpha)$.*

Proof. Let $R^{0,1}$ be R for ease of notation and let P be the recursive relation described in Lemma 3. We first define R' as:

$$R'(\alpha) \leftrightarrow \forall\beta\exists m\forall n\forall k[P(n,k,\alpha,\beta) \to P(n,m,\alpha,\beta)] \tag{1}$$

Let f and g be the functions described in Lemma 1 and 2 respectively. Let $\mathcal{B}$ be any collection of basis elements of $\mathcal{T}$, with $\beta = g(\mathcal{B})$. Suppose first that $\mathcal{B}$ does not form a covering of $\mathcal{V}$. By Lemma 3, it follows that for all n and k, $P(n,m,\alpha,\beta)$ is false and the matrix in the right hand side of Eq. 1 holds true vacuously. So suppose that $\mathcal{B}$ forms a covering of $\mathcal{V}$. By assumption, there exists a finite collection of basis elements $\mathcal{F} \subseteq \mathcal{B}$ that forms a covering of $\mathcal{V}$. Since $|\mathcal{F}|$ is finite, we have $\mathcal{F} \subseteq \mathcal{G}$ for some finite $\mathcal{G}$ such that for some m, $\mathcal{G}$ corresponds to $\overline{\beta}(m)$. It follows that $\mathcal{G}$ is likewise a finite covering for $\mathcal{V}$ consisting of basis elements. Since $\mathcal{G}$ is a finite covering of $\mathcal{V}$, it follows that for any n, we have $P(n,m,\alpha,\beta)$ - as the finite collection of basis elements $\mathcal{G}$ that corresponds to $\overline{\beta}(m)$ covers the union of closed intervals corresponding to $\overline{\alpha}(n)$ for any $n \ge 0$. Therefore $R'(\alpha)$.

Let ψ be a pairing function $\psi: \omega \times \omega \to \omega$. Define $\gamma: \omega \times \omega \times \omega \to \omega$ to be the function $\gamma(a,b,c) := \psi(a, \psi(b,c))$. Given any tuple $x = (a,b)$, let $(x)_0 = a$ and $(x)_1 = b$. Similarly, given $x = (a,b,c)$, let $(x)_0 = a$, $(x)_1 = b$ and $(x)_2 = c$. Applying the stated result on quantifier contraction and basic results on logic, we have the following set of equivalences from which we conclude that $R(\alpha) \in \Pi^1_1$.

$$R'(\alpha) \leftrightarrow \forall\beta\exists m\forall n\forall k[P(n,k,\alpha,\beta) \to P(n,m,\alpha,\beta)]$$
$$\leftrightarrow \forall\beta\exists m\forall z[P((\psi^{-1}(z))_0, (\psi^{-1}(z))_1, \alpha, \beta) \to P((\psi^{-1}(z))_0, m, \alpha, \beta)]$$
$$\leftrightarrow \forall\beta\exists z[P((\gamma^{-1}(z))_1, (\gamma^{-1}(z))_2, \alpha, \beta) \to P((\gamma^{-1}(z))_1, (\gamma^{-1}(z))_0, \alpha, \beta)]$$

Let $\alpha = f(\mathcal{V})$ and let $\beta = g(\mathcal{B})$ for some collection of basis elements $\mathcal{B}$. Let $R'(\alpha)$ be defined as in Eq 1. By the result on quantifier contraction, R' is equivalent to some $R \in \Pi^1_1$ with the rightmost universal quantifier (over numbers) removed. By assumption, we have $R(\alpha)$ and therefore $R'(\alpha)$. By definition of R', we have that there exists an m such that for all n and k, $P(n,k,\alpha,\beta) \to P(n,m,\alpha,\beta)$ is true. If given n and k, $P(n,k,\alpha,\beta)$ is false, then the collection of basis elements corresponding to $\overline{\beta}(k)$ does not form a covering of $\mathcal{V}$ so that the proof holds vacuously. If given n and k, $P(n,k,\alpha,\beta)$ is true, by Lemma 3, the finite set of basis elements corresponding to $\overline{\beta}(k)$

forms a covering of the union of closed intervals corresponding to $\overline{\alpha}(n)$. But since R' is true, for all β, there exists an m such that the finite set of basis elements corresponding to $\overline{\beta}(m)$ also forms a covering of the union of closed intervals corresponding to $\overline{\alpha}(n)$, i.e., $P(n,m,\alpha,\beta)$ is true (following Lemma 3). As this m holds for all n and k, we have that the finite set of basis elements corresponding to $\overline{\beta}(m)$ is the desired finite subcovering consisting of basis elements. ■

Theorem 3 *There exists a relation $R^{0,1} \in \Pi_1^1$ such that for any $\mathcal{V} \in V$ with $\alpha = f(\mathcal{V})$, we have:*

$$\mathcal{V} \text{ is compact} \leftrightarrow R^{0,1}(\alpha)$$

Proof. Let $R^{0,1}$ be R for ease of notation. Define $R(\alpha)$ such that $R(\alpha) \leftrightarrow R'(\alpha)$, where $R'(\alpha) \in \Pi_1^1$ is described in Theorem 2. Thus, $R(\alpha) \in \Pi_1^1$. To prove the ($\leftarrow$) direction, suppose $R(\alpha)$ and therefore $R'(\alpha)$. By Theorem 2, it follows that for each covering of $\mathcal{V}$ by a collection of basis elements, there exists an m such that $\overline{\beta}(m)$ forms a finite subcollection of basis elements that also covers $\mathcal{V}$. To show that $\mathcal{V}$ is compact, by a standard result of general topology regarding basis, any collection $\mathcal{O}$ of open sets is mapped to a corresponding collection $\mathcal{B}$ of basis elements such that for all $O \in \mathcal{O}$ and all $x \in O$, there exists a $B \in \mathcal{B}$ such that $x \in B$ (call this the equivalence conditions between $\mathcal{O}$ and $\mathcal{B}$). Using contraposition, suppose that $\mathcal{V}$ is not compact, then there exists a covering $\mathcal{O}$ such that no finite subcovering of $\mathcal{O}$ covers $\mathcal{V}$. Let $\mathcal{B}$ be the set of basis elements satisfying the equivalence conditions with $\mathcal{O}$, plus the additional condition that each $B \in \mathcal{B}$ is mapped to a corresponding $O \in \mathcal{O}$ such that $B \subseteq O$. By definition of basis, such a $\mathcal{B}$ exists. It follows that for $\beta = g(\mathcal{B})$ (using the function g described in Lemma 2), there does not exist an m whereby $P(n,m,\alpha,\beta)$ holds for any n. This implies $\neg R'(\alpha)$ and therefore $\neg R(\alpha)$.

Alternatively, for a direct proof, suppose that for each cover of $\mathcal{V}$ by basis elements, there exists a finite subcover consisting also of basis elements. We show that $\mathcal{V}$ is compact. Let $\mathcal{O}$ be any covering of $\mathcal{V}$, and let $\mathcal{B}$ be the collection of basis elements that satisfy the equivalance conditions with $\mathcal{O}$, plus the additional condition that for each $O \in \mathcal{O}$, there exists a $B \in \mathcal{B}$ such that $B \subseteq O$. It is possible to derive such a collection $\mathcal{B}$ by definition of a basis. Given that $\bigcup \mathcal{O} = \bigcup \mathcal{B}$, it follows that $\mathcal{B}$ forms a covering of $\mathcal{V}$. Since $R'(\alpha = f(\mathcal{B}))$ is true, by Theorem 2, there exists a finite subcollection $\mathcal{F} \subseteq \mathcal{B}$ that also covers $\mathcal{V}$, such that given $\beta = g(\mathcal{B})$, we have that $\mathcal{F}$ corresponds to $\overline{\beta}(m)$ for some m. Let $\mathcal{F} = \{B_0, B_1, \dots, B_d\}$ for some $d < m$. Given that for $0 \leq i \leq d$, we have $B_i \subseteq O_i$ for some $O_i \in \mathcal{O}$, we can form the desired finite collection of open sets $\{O_0, O_1, \dots, O_d\} \subseteq \mathcal{O}$ which also covers $\mathcal{V}$. Since this holds true for any covering $\mathcal{O}$ of $\mathcal{V}$, $\mathcal{V}$ is therefore compact.

To prove the ($\rightarrow$) direction, suppose that $\mathcal{V}$ is compact. Since $\mathcal{V}$ is compact, any covering $\mathcal{O}$ of $\mathcal{V}$ has a finite subcollection $\mathcal{F} \subseteq \mathcal{O}$ that also covers $\mathcal{V}$. Let $\mathcal{B}$ be a covering for $\mathcal{V}$ consisting of a collection of basis elements. Since $\mathcal{V}$ is compact and $\mathcal{B}$ is a covering, there exists finite subcollection $\mathcal{F} \subseteq \mathcal{B}$ that covers $\mathcal{V}$. It follows that $\mathcal{F} \subseteq \mathcal{G}$ for some finite subcollection $\mathcal{G}$, such that given $\beta = g(\mathcal{B})$, $\mathcal{G}$ corresponds to $\overline{\beta}(m)$ for some m. Since this holds true for any collection $\mathcal{B}$ of basis elements that covers $\mathcal{V}$, it follows that for any $\beta = g(\mathcal{B})$, there exists a finite subcover $\mathcal{G} \subseteq \mathcal{B}$ that corresponds to $\overline{\beta}(m)$ for some m - such that for all n, $P(n,m,\alpha,\beta)$ holds true. Therefore $R'(\alpha)$ and equivalently, $R(\alpha)$. ■

Corollary 1 *Given any $\mathcal{V} \in V$, let $\alpha = f(\mathcal{V})$. Let R be the relation described in Theorem 3. Then $R \ll W$.*

Proof. Follows readily from Theorems 1 and 3. ■

Corollary 2 *Let $\mathcal{V} \in V$ be compact, and let h be any continuous function $h: P(X) \rightarrow P(X)$. Let $\alpha = f(h(\mathcal{V}))$. Let R be the relation described in Theorem 3. Then $R(\alpha)$.*

Proof. Given that $\mathcal{V}$ is compact and countable, we have that $h(\mathcal{V})$ is also compact - using the result that continuous images of compact spaces are also compact. By the Heine-Borel theorem, compact spaces of X are closed and bounded (Munkres 2000). It follows that $h(\mathcal{V})$ is likewise closed and bounded, i.e., made up of a countable collection of closed intervals (a point is also considered a closed interval and functions map countable sets to countable sets). Thus, given $\alpha = f(h(\mathcal{V}))$, by Theorem 3, $R(\alpha)$. ■

CONCLUSION AND RECOMMENDATIONS

In this paper, we showed that the descriptive complexity of a compact countable collection of sets that are formed from unions of closed intervals of real numbers belongs to Π_1^1. To prove this result, we defined a recursive R such that for any function α that corresponds to the special types of collection of sets just described, we have that the collection is compact if and only if $R(\alpha)$ holds.

A natural extension of this paper is to inquire whether given the R described in Theorem 3, there is a mapping $W \ll R$. If this is true, it follows that the descriptive complexity of evaluating a set in Π_1^1 is akin to evaluating the compactness of a collection of sets belonging to V. It is to be noted that this paper only considers compactness of collections of sets belonging to V. However the topological space in which V is found is metrizable and as such, satisfies nice properties such as separability which facilitates analysis of its descriptive complexity. A natural extension is to consider if it is viable to evaluate the descriptive complexities of more arbitrary collections of sets under weaker topological spaces, i.e., spaces that are not metrizable or spaces that follow weaker Axioms such as the T1 Axiom whereby for any two distinct points, each point has a neighborhood that does not contain the other point. Possible lack of separability in these spaces along with the arbitrary cardinality of its collection of basis elements may not guarantee that the collection of compact sets in every instance of these spaces has an effective description.

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

CONTRIBUTIONS OF INDIVIDUAL AUTHORS

The first author worked on drafting the paper's results and proofs. The second author provided topic conceptualization, ample guidance and review of theoretical results.

REFERENCES

Chong, CT, Yu L. Recursion theory: Computational aspects of definability. Walter de Gruyter GmbH & Co KG 2015; 8.

Forster Y, Kunze F, & Wuttke M. Verified programming of turing machines in coq. Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs 2020; 114–128.

Herrlich H. Topological Functors. General Topology and Its Applications 1974; 4:125-142.

Hinman PG. Recursion-theoretic hierarchies. Cambridge University Press 2017; 9.

Kechris A. Classical descriptive set theory. Springer Science & Business Media 2012; 156.

Lutzer D, van Mill J, & Pol R. Descriptive Complexity of Function Spaces. Transactions of the American Mathematical Society 1985; 291 (1): 121–128. https://doi.org/10.2307/1999898

Matheron E. The Descriptive Complexity of Helson Sets. Illinois Journal of Mathematics 1995; 39 (4).

Moschovakis YN. Descriptive set theory. American Mathematical Soc 2009.

Moschovakis YN. Classical descriptive set theory as a refinement of effective descriptive set theory. Annals of Pure and Applied Logic 2010; 162 (3): 243–255.

Munkres JR. Topology. Prentic Hall of India Private Limited, New Delhi 2000; 7.

Scott D. Lectures on a mathematical theory of computation. Technical Monograph PRG-19. Oxford Univ. Comp. Lab 1981.

Spreen D. A characterization of effective topological spaces. Recursion Theory Week. Lecture Notes in Mathematics, Springer, Berlin, Heidelberg 1990. 1432:363–387.

Yu L. An application of recursion theory to analysis. Bulletin of Symbolic Logic 2020; 26 (1):15–25.9